DE TRANSFORMATIONE AEQUATIONIS

$$y^n = R(x),$$

DESIGNANTE $R(x)$ FUNCTIONEM INTEGRAM RATIONALEM VARIABILIS x,

IN AEQUATIONEM

$$\eta^2 = R_1(\xi).$$

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§ 1.

Data aequatione $f\,(x\,y) = 0$ inter x et y, si formamus functionem rationalem $F(xy)$, haec pro numero finito $p+1$ valorum x et y, ex. gr. pro $x_0 y_0$, $x_1 y_1 \ldots x_p y_p$ fit infinita, dicemusque, esse $F(xy)$ ordinis $p+1$. Est autem pro numero p limes inferior ϱ, ita ut generaliter non possit formari functio, quae pro minus quam $\varrho + 1$ valoribus, ad arbitrium sumptis fiat infinita. Qui numerus ϱ e sola functionis $f(xy)$ natura aptum esse scimus. Quem invenimusque hac ratione.

Sit aequatio $f(xy) = 0$ ordine m respectu x, ordine n respectu y; sint a_1, $a_2 \ldots a_q$ valores illi singulares ipsius x, pro quibus concidunt aut omnes aut nonnulli ex n valoribus variabilis y, qui sunt generaliter diversi; sintque repraesentationes y pro $x = a_\lambda$ divisi in g_λ series, quarum μ^{ta} continet $n_{\lambda\mu}$ valores, ita ut sit $\Sigma_\mu\, n_{\lambda\mu} = n$.

Quibus rebus constitutis demonstrari potest, discriminantem functionis cujusvis $F(xy)$ inter x et y, habere factorem alterum $D_1 = \Pi\,(x - a_\lambda)^{\,n - g_\lambda}$, cui a viro Ill. Kronecker, qui primus tractavit has rationes nomen inditum est „factor essentialis" (wesentlicher Factor), propterea quod tantum dependet ab aequatione $f(xy) = 0$, minime autem ab aequatione $F(xy) = 0$; alterum factorem, quem nominat „factorem extraessentialem" (außerwesentlicher Factor) cum aptus sit et is ex aequatione $F(xy) = 0$. Jam vero si significamus litera s ordinem D_1, et litera t ordinem D_2, demonstrantur hae relationes

$$2\,(n-1)\,m = s + 2\,t; \qquad s = \Sigma_\lambda\,(n - g_\lambda).$$

Deinde definitur ϱ aequationibus

$$\varrho = (m-1)\,(n-1) - t = \frac{s}{2} - (n-1);$$

(1)

$$\varrho = \frac{1}{2}\,\Sigma_\lambda\,(n - g_\lambda) - (n-1);$$

quod ϱ transformatione uniformi aequationis $F(xy)$ non mutari scimus.

Adhibeamus nunc theoriam hanc generalem ad perquirendam functionem

(2) $$f(xy) = y^n - (x - a_1)^{m_1}\,(x - a_2)^{m_2} \ldots\ldots (x - a_r)^{m_r} = 0,$$

in qua nullus inter n et omnia m divisor sit communis; ponamus insuper

$$(3)\qquad \begin{aligned}&(x-a_1)=(1),\ (x-a_2)=(2),\ \ldots.\\&(x'-a_1)=(1'),\ (x'-a_2)=(2'),\ \ldots.\end{aligned}$$

atque

$$(3')\qquad \begin{aligned}m_1&=g_1\,\mu_1;\ m_2=g_2\,\mu_2;\ \ldots.\ m_r=g_r\,\mu_r,\\ n&=g_1\,\nu_1\quad =g_2\,\nu_2\ \ldots.\ =g_r\,\nu_r,\end{aligned}$$

ubi μ_λ et ν_λ sunt nnmeri inter se primi. Sed ut faciliorem reddamus disquisitionem, facta transformatione $x=\frac{\xi-b}{\xi-d}$, expressionem

$$y^n=C\frac{(\xi-c_1)^{m_1}(\xi-c_2)^{m_2}\ldots.(\xi-c_r)^{m_\nu}}{(\xi-d)^{\Sigma m}}$$

multiplicemus per $\frac{(\xi-d)^{np-\Sigma m}}{(\xi-d)^{np-\Sigma m}}$ (ubi $np-\Sigma m<n$ esse oportet), ac ponamus $\eta=y.(\xi-d)^p$. Qua ratione efficitur, ut summa exponentium m, per n dividi possit, i. e. ne sit punctum $x=\infty$ singulare.

Ac consideramus quidem aequationem (2) ut transformatam, in qua est $\Sigma m=np$. Patet puncta $a_1, a_2 \ldots a_r$ esse singularia. Nunc nobis videtur disquirenda esse repraesentatio functionis y pro valore x prope a_λ. Ad quod perficiendum si ponitur $x-a_\lambda=u_\lambda$, est

$$y^n=u_\lambda^{m_\lambda}R_1(x)=R_1(a_\lambda)\,u_\lambda^{m_\lambda}(1+\alpha_1 u_\lambda+\alpha_2 u_\lambda^2+..),$$

atque ponamus $$c_\lambda^{g_\lambda}=R_1(a_\lambda);$$

deinde cum μ_λ et ν_λ sint numeri inter se primi (3′):

$$\mu_\lambda\,\sigma_\lambda-\nu_\lambda\,\tau_\lambda=1.$$

Unde sequitur
$$\begin{aligned}R_1(a_\lambda)\,u_\lambda^{m_\lambda}&=c_\lambda^{g_\lambda(\mu_\lambda\sigma_\lambda-\nu_\lambda\tau_\lambda)}\,u_\lambda^{g_\lambda\mu_\lambda}\\&=(c_\lambda^{\sigma_\lambda}u_\lambda)^{m_\lambda}\,c_\lambda^{-g_\lambda\nu_\lambda\tau_\lambda},\end{aligned}$$

atque si significabimus

$$(4)\qquad x-a_\lambda=u_\lambda=c_\lambda^{-\sigma_\lambda}\,t^{\nu_\lambda}$$

vel
$$t=(u_\lambda\,c_\lambda^{\sigma_\lambda})^{\frac{1}{\nu_\lambda}}$$

erit

$$(5)\qquad y=c_\lambda^{-\tau_\lambda}\,t^{\mu_\lambda}(1+\alpha_1\,c^{-\sigma_\lambda}\,t^{\nu_\lambda}+..)^{\frac{1}{n}}.$$

Cum vero t valoribus utatur ν_λ, c_λ autem g_λ, repraesentationes functionis y prope $x=a_\lambda$ dirimuntur in g_λ partes, quarum quaeque continet repraesentationes ν_λ; ergo omnino habemus $\sum_\mu g_\lambda\,\nu_\lambda=n$, id quod valere necesse est.

Unde sequitur, ut pro puncto a_λ in discriminante existat factoris essentialis divisor hic: $(x-a_\lambda)^{n-g_\lambda}$, idemque fiet in reliquis punctis singularibus.

Quam ob rem valor est ipsius s

(6) $$s = \sum_{\lambda} (n - g_\lambda) = nr - \sum_{\lambda=1}^{r} g_\lambda,$$

atque inde $$2\varrho = nr - \sum_{\lambda} g_\lambda - 2n + 2,$$

vel

(7) $$2(\varrho - 1) = n(r-2) - \sum_{\lambda=1}^{r} g_\lambda.$$

§ 2.

Jam vero id nobis agendum videtur esse, ut rem inversam tractemus, i. e. ut constituamus eas functiones $f(xy)$ formae (2), quae pertinent ad ϱ quoddam datum.

Qua in re hac utimur ratione, ut ipsius r, (qui numerus indicat factorum $R(xy)$ diversorum multitudinem) limitem et superiorem et inferiorem literae n definiamus.

Ac primum quidem intelligitur, id quod formula 7 paragr. 1 demonstrat, si ponitur $\varrho > 0$, non posse esse $r = 2$.

Tum si est $r = 3$, sequitur
$$2(\varrho - 1) = n - (g_1 + g_2 + g_3).$$
Quae g_1, g_2, g_3, cum sint divisores ipsius n, maximum esse n apparet, si sint illa g divisores quam maximi. Quod extemplo sequitur, si latus dexterum redigitur in formam
$$n\left(1 - \left[\frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3}\right]\right).$$
Ac si n est par, g_1 maxime est $\frac{n}{2}$.

Cum vero $\mu_1 g_1 + \mu_2 g_2 + .. + \mu_r g_r = n.p$ sit, nec omnes r, nec $r-1$ ex illis g divisorem communem habere licet, quoniam radix reduci posset, hoc igitur loco ne duo quidem. Quoniam autem $\frac{n}{2}$ divisores ipsius n continet omnes excepto divisore 2, si est $n = 4.q + 2$, non possunt locum habere sicut valores maximi, nisi
$$g_1 = \frac{n}{2},\ g_2 = 2,\ g_3 = 1,$$
atque erit
$$2(\varrho - 1) = n - \left(\frac{n}{2} + 2 + 1\right) = \frac{n}{2} - 3;$$
$$n \leqq 2(2\varrho + 1).$$
Nunc si n est impar, simili modo efficitur, ut illorum g valores summi inveniantur
$$g_1 = \frac{n}{3},\ g_2 = \frac{n}{5},\ g_3 = \frac{n}{7}$$

unde $$2(\varrho-1)=n-\left(\frac{n}{3}+\frac{n}{5}+\frac{n}{7}\right)=\frac{34}{105}n$$

$$n\leqq\frac{105}{17}(\varrho-1)\leqq 7(\varrho-1)-2.$$

Quae quidem formula si est $\varrho=1$ non valet, cum pro $n=3$, $g_1=g_2=g_3=\frac{n}{3}=1$ esse possit. Quamquam hic quae omnino extant formae quatuor protinus intelliguntur:

$$\varrho=1$$

1. $y=\sqrt[6]{(1)(2)^2(3)^3}$; 3. $y=\sqrt[3]{(1)(2)(3)}$;

2. $y=\sqrt[4]{(1)(2)(3)^2}$; 4. $y=\sqrt[2]{(1)\,..\,(4)}$.

Viri Ill. Briot et Bouquet, qui in libro: „Théorie des fonctions doublement périodiques" de hoc casu disserunt, cum nec illa quam exhibuimus simplificatione nec altera postea definienda utantur, 11 formas diversas afferunt.

Deinde si est $r\gtreqless 4$, iterum disquirendum videtur, quid sit n, si est vel par vel impar. Ac si illud fit, invenitur:

$$g_1=g_2=\ldots=g_{r-2}=\frac{n}{2};\ g_{r-1}=g_r=2.$$

$$2(\varrho-1)=n(r-2)-(r-2)\frac{n}{2}-4=\frac{n(r-2)}{2}-4,$$

$$n\leqq\frac{4(\varrho+1)}{r-2};$$

sin hoc,

$$g_1=g_2=\ldots=g_{r-2}=\frac{n}{3};\ g_{r-1}=g_r=\frac{n}{5}$$

$$2(\varrho-1)=n(r-2)-(r-2)\frac{n}{3}-\frac{2n}{5}=\frac{2n}{15}(5r-13),$$

$$n\leqq\frac{15(\varrho-1)}{5r-13}<\frac{3(\varrho-1)}{r-3}.$$

Limes autem inferior illorum n sequitur ex hoc, quod patet g illa minimum esse $=1$. Ergo

$$n(r-2)-r=2(\varrho-1)$$

$$n\geqq\frac{2(\varrho-1)+r}{r-2}=1+\frac{2\varrho}{r-2}.$$

Qua ratione quaecunque extant n atque quae ad illa pertinent g inventis, restat ut ipsa $m_1, m_2 \ldots m_r$ constituantur. Solutionum tamen numerus transformatione simplici et superioris simili aliquanto diminuitur. Etenim si iterum nos applicamus ad aequationem

$$y^n=(1)^{m_1}(2)^{m_2}\ldots(r)^{m_r}$$

eamque elevabimus ad potestatem q^{tam}, ubi q et n sunt numeri inter se primi, si est

$$m_1q=nk_1+m_1',\ (m_1'<n)$$

$$m_2q=nk_2+m_2',\ (m_2'<n)$$

. . . .

inveniemus $$\left\{\frac{y}{(1)^{k_1}\dots(r)^{k_r}}\right\}^n = y'^n = (1)^{m'_1}\dots(r)^{m'_r},$$

quae forma virtutibus praedita est iisdem, quibus superior. Nam quoniam

$$q\sum_{\lambda=1}^{r} m_\lambda = n\sum_{\lambda=1}^{r} k_\lambda + \sum_{\lambda=1}^{r} m'_\lambda,$$

$\sum m'_\lambda$ dividi potest per n; cumque in

$$m_\lambda q = nk_\lambda + m'_\lambda$$

m_λ et n maximo divisore communi utantur g_λ, m'_λ eodem utitur.

Commemoramus denique, pro ϱ dato semper extare formam

$$y^2 = (1)(2)\dots(2\varrho+2).$$

Enumeremus nunc, quod jam fecimus pro $\varrho = 1$, pro $\varrho = 2$ et $\varrho = 3$ formulas $y^n = R(x)$ inventas ratione illa modo explicata.

$\varrho = 2.$

1. $y^{10} = (1)\ (2)^4\ (3)^5,$ $\qquad r = 3$
2. $y^8 = (1)\ (2)^3\ (3)^4,$
3. $y^6 = (1)\ (2)\ (3)^4,$
4. $y^5 = (1)\ (2)\ (3)^3;$
5. $y^6 = (1)^2\ (2)^3\ (3)^3\ (4)^4;$ $\qquad r = 4$
6. $y^4 = (1)\ (2)^2\ (3)^2\ (4)^3,$
7. $y^3 = (1)\ (2)\ (3)^2\ (4)^2;$
8. $y^2 = (1)\ (2)\dots(6).$ $\qquad r = 6.$

$\varrho = 3.$

1. $y^{14} = (1)^2\ (2)^5\ (3)^7,$ $\qquad r = 3$
2. $y^{12} = (1)\ (2)^5\ (3)^6,$
3. $y^{12} = (1)^3\ (2)^4\ (3)^5,$
4. $y^9 = (1)\ (2)^3\ (3)^5,$
5. $y^8 = (1)\ (2)^2\ (3)^5,$
6. $y^7 = (1)\ (2)\ (3)^5;$
7. $y^6 = (1)\ (2)^3\ (3)^4\ (4)^4,$ $\qquad r = 4$
8. $y^6 = (1)\ (2)^3\ (3)^3\ (4)^5,$
9. $y^4 = (1)\ (2)\ (3)\ (4);$
10. $y^4 = (1)\ (2)\ (3)^2\ (4)^2\ (5)^2,$ $\qquad r = 5$
11. $y^3 = (1)\ (2)\ (3)\ (4)\ (5)^2;$
12. $y^2 = (1)\ (2)\ \dots\ (8).$ $\qquad r = 8.$

§ 3.

Qua ratione methodis explicatis, quibus fieri possit, ut omnes formae $y^n = R(x)$ ad ϱ quoddam datum pertinentes constituantur, cum videamus, ad omnes valores ϱ pertinere formam $y^2 = R_1(x)$, quaerere liceat, quae inter formas inventas transformatione uniformi mutari possint in $\eta^2 = R_1(\xi)$. Quam ad rem opus videtur esse paucis exponere illa, quae in substitutionum theoria inventa sunt.

Ac primum, ut ordo formae

(1) $$\xi = P(xy)$$

constituatur, formetur expressio

(2) $$R(\xi x) = [\xi - P(xy_1)]\,[\xi - P(xy_2)] \ldots . [\xi - P(xy_n)],$$

in qua valores $y_1 \ldots . y_n$ pertinent ad x secundum aequationem datam $f(xy) = 0$. Est $R(\xi x)$ rationale in x et ξ atque si indice minimo generali singulorum coefficientium potestatum ξ multiplicatur $R(\xi x)$, haec expressio ordinem functionis ξ indicat, sive $R(\xi x)$ reductibile est sive irreductibile (qua in re dicendum est, illud quidem semper vitari posse). Sed quod haec conclusio nititur eo, quod productum (2) fit 0, si quis factor est 0, non valet pro $\xi = \infty$, $P(xy_\lambda) = \infty$, in quo casu plures factores erunt ∞. Attamen si consideramus ordinem functionis $\frac{1}{P(xy)}$, apparet eundem esse ac functionis $P(xy)$, et cum functio illa sit $= 0$ pro eodem numero valorum, quo habeat valorem finitum quemcunque, haec pro eodem numero erit ∞.

Sed si consideramus exempli gratia

$$\xi = \frac{1}{\sqrt{(x-a)(x-b)}},$$

quae functio est secundi ordinis quoniam pro $x = a$, $x = b$ fit ∞, tamen haec pro his valoribus habet repraesentationem

$$\xi = (x-a)^{-\frac{1}{2}} + \ldots .,$$

ita ut $[(x-a)\xi]_{x=a}$ sit 0. Dicemus igitur, id quod magnopere simplicabit expressionem nostram „ξ, quamquam appropinquat ad valorem infinitum, tamen non esse infinitum", vel „esse $(x-a)^{-\varepsilon}$ pro $x = a$ si $0 < \varepsilon < 1$, egale ∞^ε i. e. esse infinitum minoris quam primi ordinis".

Inter quas formas $\xi = P(xy)$ eae maximi momenti sunt, quarum functio integralis pro ullo valore variabilium x et y vel finito vel infinito non fiat $= \infty$. Quam ad rem perficiendam facile intelligitur necesse esse, ut hae formae, quae auctore viro Ill. Weierstraſs significentur $H(xy)$, pro nullo valore finito fiant ∞ primi ordinis (cum ∞^ε, $0 < \varepsilon < 1$ fieri possint), pro $x = \infty$ vero fiant 0 altioris quam primi ordinis. Est igitur

(3) $$\begin{aligned} &[(x-a)\,H(xy)]_{x=a} = 0, && \text{si } a \text{ est finitum;} \\ &x\,H(xy) = 0, && \text{si } x \text{ est infinitum.} \end{aligned}$$

Qualium $H(xy)$ ad $f(xy) = 0$ pertinentium numerum egalem esse ϱ (aequatione nulla lineari

(4) $$c_1 H(xy)_1 + c_2 H(xy)_2 + \ldots + c_\varrho H(xy)_\varrho = 0$$

extante), atque ex tali systemate functionum fundamentalium $H(xy)_1 \ldots H(xy)_\varrho$ omne aliud $H(xy)$ generale lineariter componi posse, satis notum est.

Qua in re hoc afferendum est, quod nonnumquam efficit, ut omnes ϱ functiones $H(xy)$ simpliciore modo ex quibusdam inter illas reputentur. Etenim productum ex q functionibus $H(xy)$ formatum maxime ordinis $q - 1 + \varepsilon$ infinitum potest fieri. Itaque radix q^{ta} maxime $\frac{q-1+\varepsilon}{q} < 1$ ordinis fit infinita; cum sit deinde $x H(xy)_\lambda = 0$ pro $x = \infty$, erit item $x^q \prod_{\lambda=1}^{q} H(xy)_\lambda = 0$ ac $x \sqrt[q]{\prod_{\lambda=1}^{q} H(xy)_\lambda} = 0$ pro $x = \infty$. Si igitur fieri potest, ut haec radix q^{ta} in x et y praebeatur rationalis, rursus $H(xy)$ quoddam reperitur; qua de causa in hoc casu radix q^{ta} quoque lineariter potest exprimi per ϱ functiones fundamentales $H(xy)$. Quod idem fieri intelligitur, si productum supplemus functione integra homogeni ordinis q^{ti} functionum $H(xy)_1, \ldots H(xy)_\varrho$. Cujus rei exempla in ratiocinandis $H(xy)$ illis postea se praebebunt. — Sed haec hactenus.

Quod vero ad transformationem attinet, ponatur:

(5) $$\xi = P(xy); \quad \eta = Q(xy);$$

et id $P(xy)$ quidem sit ordinis g_1, $Q(xy)$ autem ordinis h_1. Si valores $\eta_1, \eta_2, \ldots \eta_g$, qui attinent ad valores $x_1 y_1, x_2 y_2, \ldots x_{g_1} y_{g_1}$ sunt diversi inter se, cum paria $x_1 y_1, x_2 y_2, \ldots, x_{g_1} y_{g_1}$ idem efficiant ξ; x et y rationaliter exprimi possunt per ξ et η, nempe

(6) $$x = P_1(\xi\eta); \quad y = Q_1(\xi\eta).$$

Quibus expressionibus $f(xy) = 0$ in $\varphi(\xi\eta) = 0$ transformatur, et formulis (5) contra $\varphi(\xi\eta) = 0$ in $f(xy) = 0$.

Erit autem $\varphi(\xi\eta) = 0$ respectu ξ ordinis h^{ti}, et respectu η ordinis g^{ti}. Quae transformatio uniformis est, i. e. unicuique pari xy, unum respondet par $\xi\eta$; idem etiam contra valet. In casu igitur a nobis requisito $P(xy) = \xi$ sit ordinis secundi necesse est, ut possit effici transformatio formae $f(xy) = y^n - R(x) = 0$ in $\varphi(\xi\eta) = \eta^2 - R_1(\xi) = 0$.

§ 4.

Ac primum quidem, cum in posterum formae $H(xy)$ adhibeantur, id nobis agendum est, ut in casu a nobis accuratius exquisito

(1) $$y^n = R(x) = (1)^{m_1} (2)^{m_2} \ldots (r)^{m_r}$$

ipsae illae functiones exhibeantur.

Expressio autem generalis omnis functionis rationalis variabilium x et y est

$$F(xy) = \frac{f_0(x) + f_1(x)y + f_2(x)y^2 + \ldots + f_{n-1}(x)y^{n-1}}{y^{n-1}} \tag{2}$$

in qua $f_0(x)$, $f_1(x)$ .... $f_{n-1}(x)$ illa functiones sunt solius variabilis x. Fac $F(xy)$ esse $H(xy)$ aliquod, necesse est $F(xy_1)$ .... $F(xy_n)$ manere functiones ejusdem generis, si y_1, y_2 .... y_n illa sunt y, quae ad valorem quendam variabilis x pertinent. Quae cum in nostro casu radicibus unitatis differant, sequitur, si ε est radix primitiva aequationis $x^n = 1$,

$$\begin{aligned} F(xy_1) &= \frac{f_0(x) + f_1(x)y + f_2(x)y^2 + \ldots + f_{n-1}(x)y^{n-1}}{y^{n-1}} \\ F(xy_2) &= \frac{f_0(x)\varepsilon + f_1(x)y\varepsilon^2 + f_2(x)y^2\varepsilon^3 + \ldots + f_{n-1}(x)y^{n-1}}{y^{n-1}} \\ &\ldots\ldots\ldots\ldots \\ F(xy_n) &= \frac{f_0(x)\varepsilon^{n-1} + f_1(x)y\varepsilon^{2(n-1)} + f_2(x)y^2\varepsilon^{3(n-1)} + \ldots + f_{n-1}(x)y^{n-1}}{y^{n-1}} \end{aligned} \tag{3}$$

ita ut

$$F(xy_1) + \frac{1}{\varepsilon^{\lambda+1}} F(xy_2) + \frac{1}{\varepsilon^{2(\lambda+1)}} F(xy_3) + \ldots = \frac{f_\lambda y^\lambda}{y^{n-1}}. \tag{4}$$

Quae expressio (4) item est $H(xy)$ aliquod, sintque hae expressiones ejus formae functiones $H(xy)$ fundamentales. Nec vero $f_\lambda(x)$ illa possunt esse functiones fractae. Nam si quis factorum $(x-a)^p$, qui idem reperitur in $R(x)$, in indice esset positus, non fieret

$$\left(\frac{f_\lambda(x)}{y^{n-\lambda-1}}(x-a)\right)_{x=a} = 0,$$

sed fieret ∞; atque idem esset pro factore aliquo indicis $(x-b)^q$, quem non continet $R(x)$. Dividamus igitur $f_\lambda(x)$ in duas partes, ut in altera tantum factores (1), (2) .... (r); in altera tantum factores alieni $(x-b_1)$ $(x-b_2)$ .... collocentur, ita ut sit:

$$f_\lambda(x) = (1)^{m_1^{(\lambda)}} (2)^{m_2^{(\lambda)}} \ldots (r)^{m_r^{(\lambda)}} \prod (x-b).$$

Esto $m_1^{(\lambda)} + m_2^{(\lambda)} + \ldots + m_r^{(\lambda)} = m^{(\lambda)}$ atque h ordo producti $\prod(x-b)$. Siquidem pro $x = \infty$ opus est esse $\frac{f_\lambda(x).x}{y^{n-\lambda-1}} = 0$, existit aequatio conditionalis:

$$m^{(\lambda)} + h + 1 - p(n-\lambda-1) < 0,$$

ergo cum nisi de numeris integris h, p, n, λ, $m^{(\lambda)}$ non agatur

$$m^{(\lambda)} + h + 1 - p(n-\lambda-1) \leqq -1$$

$$h \leqq p(n-\lambda-1) - m^{(\lambda)} - 2. \tag{5}$$

Jam vero si conditio altera exquiritur illa pro valore finito $x = a_\lambda$, erit:

$$m_1^{(\lambda)} + 1 - \frac{1}{n}(n - \lambda - 1)\, m_1 > 0$$

(6) vel $$m_1^{(\lambda)} > \frac{m_1}{n}(n - \lambda - 1) - 1.$$

Unde sequitur, ut sit

$$m_1^{(\lambda)} + m_2^{(\lambda)} + \ldots + m_r^{(\lambda)} > \frac{m_1 + m_2 + \ldots + m_r}{n}(n - \lambda - 1) - r > p\,(n - \lambda - 1) - r;$$

ergo cum hoc loco rursus de numeris integris agatur

$$m^{(\lambda)} \geqq p\,(n - \lambda - 1) - r + 1.$$

Ex (5) autem sequitur

$$m^{(\lambda)} \leqq p\,(n - \lambda - 1) - h - 2.$$

ergo $$p\,(n - \lambda - 1) - r + 1 \leqq p\,(n - \lambda - 1) - h - 2$$

$$h \leqq r - 3.$$

Quae formula docet, nisi $R(x)$ plus tres factores inter se diversos habet, in functiones $H(xy)$ factores alienos $(x - b)$ non posse inire.

Justum autem ratiocinium fit ex formula (6), ratione quadam simplici, quae simplificatur ea re, quod vel antea numerus ϱ omnium $H(xy)$ non ignoratur.

Componamus nunc pro formis supra enumeratis (paragr. 2), quae pertinent ad valores $\varrho = 1$, $\varrho = 2$ et $\varrho = 3$ functiones $H(xy)$ inventas ea ratione, de qua modo diximus. Invenimus igitur, si numeri 1., 2. cet. respiciunt ad functiones enumeratas sub iisdem numeris extremo § 2:

$$\varrho = 1.$$

1. $$H(xy) = \frac{(2)\,(3)^2}{y^5};$$

2. $$H(xy) = \frac{(3)}{y^3};$$

3. $$H(xy) = \frac{c_1}{y};$$

4. $$H(xy) = \frac{c_1}{y}.$$

$$\varrho = 2.$$

1. $$H(xy) = (2)^2\,(3)^3\,\frac{c_1\,(2)\,(3) + c_2\,y^2}{y^9};$$

2. $$H(xy) = (2)\,(3)^2\,\frac{c_1\,(2)\,(3) + c_3\,y^2}{y^7};$$

$$3.\quad H(xy) = (3)^2 \frac{c_1 (3) + c_2 y}{y^5};$$

$$4.\quad H(xy) = (3) \frac{c_1 (3) + c_2 y}{y^4};$$

$$5.\quad H(xy) = \frac{c_1 (1)(2)^2 (3)^2 (4)^3 + c_2 y^4}{y^5};$$

$$6.\quad H(xy) = \frac{c_1 (2)(3)(4)^2 + c_2 y^2}{y^3};$$

$$7.\quad H(xy) = \frac{c_1 (3)(4) + c_2 y}{y^2};$$

$$8.\quad H(xy) = \frac{c_1 + c_2 x}{y}.$$

$\varrho = 3.$

$$1.\quad H(xy) = (2)(3)^2 \left\{\frac{c_1 (1)(2)^3 (3)^4 + c_2 (1)(2)^2 (3)^3 y^2 + c_3 y^8}{y^{13}}\right\};$$

$$2.\quad H(xy) = (2)^2 (3)^3 \left\{\frac{c_1 (2)^2 (3)^2 + c_2 (2)(3) y^2 + c_3 y^4}{y^{11}}\right\};$$

$$3.\quad H(xy) = \frac{c_1 (1)^2 (2)^3 (3)^4 + c_2 (1)(2)^2 (3)^2 y^4 + c_3 y^9}{y^{11}};$$

$$4.\quad H(xy) = (2)(3)^2 \left\{\frac{c_1 (2)(3)^2 + c_2 (2)(3) y + c_3 y^3}{y^8}\right\};$$

$$5.\quad H(xy) = (3) \frac{c_1 (2)(3)^3 + c_2 (2)(3)^2 y + c_3 y^4}{y^7};$$

$$6.\quad H(xy) = (3)^2 \frac{c_1 (3)^2 + c_2 (3) y + c_3 y^2}{y^6};$$

$$7.\quad H(xy) = \frac{c_1 (2)^2 (3)^3 (4)^3 + c_2 (2)^2 (3)^2 (4)^2 y + c_3 y^4}{y^5};$$

$$8.\quad H(xy) = \frac{c_1 (2)^2 (3)^2 (4)^4 + c_2 (2)(3)(4)^2 y^2 + c_3 y^4}{y^5};$$

$$9.\quad H(xy) = \frac{c_1 + c_2 x + c_3 y}{y^3};$$

$$10.\quad H(xy) = \frac{(2)(3)(4)(c_1 + c_2 x) + c_3 y^2}{y^3};$$

$$11.\quad H(xy) = \frac{(3)(c_1 + c_2 x) + c_3 y}{y^2};$$

$$12.\quad H(xy) = \frac{c_1 + c_2 x + c_3 x^2}{y}.$$

Videmus, in quibusdam formis pro $\varrho = 3$, sicuti in 2., 6., 8. cet., $H(xy)_2$ quod est membrum medium esse radicem ex producto $H(xy)_1$ atque $H(xy)_3$, quae sunt membra et primum et ultimum.

Quae proprietas si locum habet pro $\varrho > 3$, erit

$$H(xy)_\lambda = \sqrt[2]{H(xy)_{\lambda-1} H(xy)_{\lambda+1}},$$

atque sequitur ex hac aequatione

$$H(xy)_\lambda^4 = H(xy)_{\lambda-1}^2 H(xy)_{\lambda+1}^2 = H(xy)_{\lambda-2} H(xy)_\lambda^2 H(xy)_{\lambda+2};$$

$$H(xy)_\lambda^2 = H(xy)_{\lambda-2} H(xy)_{\lambda+2}$$

et aeque

$$H(xy)_\lambda^2 = H(xy)_{\lambda-p} H(xy)_{\lambda+p}.$$

Sequitur praeterea ex formula superiore

$$H(xy)_\lambda^4 = H(xy)_{\lambda-2} H(xy)_\lambda H(xy)_{\lambda+1}$$

$$H(xy)_\lambda^3 = H(xy)_{\lambda-2} H(xy)_{\lambda+1}^2$$

cet.

$$H(xy)_\lambda^{\varrho-1} = H(xy)_1^{\varrho-\lambda} H(xy)_\varrho^{\lambda-1}.$$

Ex quibus formulis, cum $H(xy)_1$ et $H(xy)_\varrho$ sint nota, cetera $H(xy)_\lambda$ facillime inveniri posse cognoscimus.

§ 5.

Jamjam vero in eo sumus, ut conditiones enumeremus, quibus fieri possit, ut $y^n = R(x)$ in formam $\eta^2 = R_1(\xi)$ transformetur, vel ut hujus problematis solutionem proferamus generaliorem, studeamus transformare $f(xy) = 0$ in $\varphi(\xi\eta) = 0$, ita ut sit $\varphi(\xi\eta)$ ordinis k^{ti} pro η. Quam ad rem efficiendam scimus forma $\xi = P(xy)$ inveniatur necesse esse ejusmodi, quae sit ordinis k^{ti}.

Invenit autem vir Ill. Weierstraſs functionem $H(xy, x'y')$, quae his gaudet virtutibus. Fit infinita pro $\varrho + 1$ valoribus et ea quidem primi ordinis, nempe pro $a_1 b_1$, $a_2 b_2$ .... $a_\varrho b_\varrho$; $x'y'$, — ubi $x'y'$ consideretur sicut parametros; $a_1 b_1$ .... $a_\varrho b_\varrho$ sicuti valores fixi. — Est insuper repraesentatio functionis $H(xy, x'y')$ prope $a_\lambda b_\lambda$ si ponitur $x - a_\lambda = t_\lambda$,

(1) $$H(x'y')_\lambda t_\lambda^{-1} + H^0(x'y')_\lambda - H^{(1)}(x'y')_\lambda t^1 + \ldots,$$

ubi $H(x'y')_\lambda dx'$ et $H^{(1)}(x'y')_\lambda dx'$ sunt differentiales integralium primi resp. secundi ordinis. Constat denique

$$\frac{d^\lambda H(xy, x'y')}{dx'^\lambda} = H^\lambda(xy, x'y')$$

praeterquam pro $a_1 b_1$ .... $a_\varrho b_\varrho$ ordinis primi, pro $x'y'$ ordinis $(\lambda+1)^{ti}$ fieri infinitum; atque esse repraesentationem prope $a_\lambda b_\lambda$:

(2) $$\frac{d^\lambda H(x'y')}{dx'^\lambda} t^{-1} + \frac{d^\lambda H^0(x'y')}{dx'^\lambda} - \ldots$$

Videmus ex iis, quae modo diximus, functionem $\xi = P(xy)$ ordinis k, hujus esse formae:

$$(3) \qquad P(xy) = c_1 H(xy, x_1y_1) + \dots + c_{k-1} H(xy, x_{k-1}y_{k-1}) + H(xy, x_ky_k)$$

si $x_1y_1 \dots x_ky_k$ sunt valores ii, pro quibus $P(xy) = \infty$ fit. Quod idem habet locum praeterea etiam pro ϱ valoribus $a_1 b_1, a_2 b_2 \dots a_\varrho b_\varrho$. Itaque eligenda sunt $c_1 \dots c_{k-1}$, ita ut sit

$$(4) \qquad c_1 H(x_1y_1)_\lambda + \dots + c_{k-1} H(x_{k-1}y_{k-1})_\lambda + H(x_ky_k) = 0 \quad (\lambda = 1, 2 \dots \varrho).$$

Quia autem una cum $P(xy)$ etiam $\dfrac{1}{P(xy) - P(\xi\eta)}$ functio est k^{ti} ordinis, unus ex valoribus $x_1y_1 \dots x_ky_k$ ut datus est tractandus, ex. gr. x_ky_k. Repraesentat igitur formula (3) systema ϱ aequationum inter $2(k-1)$ ignota, nempe $x_1, x_2 \dots x_{k-1}$; $c_1, c_2 \dots c_{k-1}$. Debet igitur esse:

$$(5) \qquad 2(k-1) \leqq \varrho \quad \text{vel} \quad k \leqq \frac{\varrho + 2}{2}.$$

Itaque semper formari potest functio ordinis $\dfrac{\varrho+2}{2}$ (si $\varrho = 2\alpha$), ordinis $\dfrac{\varrho+3}{2}$ (si $\varrho = 2\alpha + 1$).

Eliminatione valorum $c_1, c_2 \dots c_{k-1}$ ex (4) facta, positoque $x'y'$ in locum ipsius x_ky_k, si significatione nota utimur, habebimus conditiones

$$(6) \qquad \begin{Vmatrix} H(x_1y_1)_1, & \dots & H(x_1y_1)_\varrho \\ \vdots & & \vdots \\ H(x_{k-1}y_{k-1})_1, & \dots & H(x_{k-1}y_{k-1})_\varrho \\ H(x'y')_1, & \dots & H(x'y')_\varrho \end{Vmatrix} = 0.$$

Has conditiones quarum numerus est $\varrho - k + 1$ in aliam formam redigamus, quae valorem nullum contineat arbitrarium.

Etenim si $P(xy)$ fit ∞ pro $x_1y_1 \dots x_ky_k$ sit x_0y_0 valor singularis functionis $P(xy) - \xi$, pro quo est $P(xy) - \xi = 0$ secundi ordinis, erit $P_1(xy) = \dfrac{1}{P(xy) - P(x_0y_0)}$ functio k^{ti} ordinis, quae cum sit pro x_0y_0 infinita secundi ordinis, satis est fixa.

Ex quibus functionem nostram esse

$$(3') \qquad P_1(xy) = H(xy, x_0y_0) + c_1 H'(xy, x_0y_0) + c_2 H(xy, x_2y_2) + \dots$$

esse apparet, conditionesque quae respondent ipsis (6) erunt

$$(6') \qquad \begin{Vmatrix} H(x_0y_0)_1, & \dots & H(x_0y_0)_\varrho \\ H'(x_0y_0)_1, & \dots & H'(x_0y_0)_\varrho \\ H(x_2y_2)_1, & \dots & H(x_2y_2)_\varrho \\ \cdots & \cdots & \cdots \\ H(x_{k-1}y_{k-1})_1, & \dots & H(x_{k-1}y_{k-1})_\varrho \end{Vmatrix} = 0,$$

easdemque etiam ea obtines ratione, ut in formula (6), $x'y'$ congruere cum $x_1 y_1$ constituas.

Haud parvi mihi quidem interesse videtur dare functioni $P(xy)$ aliam formam. Inventa functione $P(xy)$ k^{ti} ordinis, formare licet

$$Q(xy) = P(xy)\left(d_1 H(xy)_1 + \dots + d_k H(xy)_k\right).$$

Factor primus fit infinitus pro $x_1 y_1 \dots x_k y_k$. Eligimus nunc $d_1 \dots d_k$ ita ut sit

$$Q(x_\lambda y_\lambda) = 0; \quad (\lambda = 1, 2 \dots k);$$

quod fieri semper potest, quoniam conditio, quae est implenda, congruit cum conditione data per aliqua formularum (6). Factor autem secundus infinitus fieri non potest, cum sit $H(xy)$ generale; nec ipsum productum infinitum erit primi ordinis, quid posset locum habere, si pro eodem valore uterque factor appropinquat ad valorem ∞. At excludamus circulis parvis puncta prope $x_1 y_1$, $x_2 y_2$, ... Quo facto terminum superiorem invenire possumus, quem $P(xy)$ excedere non potest, ac quoniam hic limes est finitus, $Q(xy)$ item semper valorem habet finitum. Cum deinde $P(xy)$ non sit ∞ pro $x = \infty$, $x H(xy)$ vero sit 0 pro hoc valore, idem locum habet pro $Q(xy)$, i. e. $Q(xy)$ est functio $H(xy)$ generalis.

Unde sequitur, ut functio quaecunque ordinis minoris quam $\varrho + 1$, quotiens sit duarum functionum $H(xy)$ generalium.

Commemoramus hoc loco etiam hanc virtutem functionum, pro quibus ϱ est egale 2. Quoniam duae tantum functiones $H(xy)$ existunt, est omne $P(xy)$ vel

$$\xi = \frac{\alpha_1 H(xy)_1 + \alpha_2 H(xy)_2}{\beta_1 H(xy)_1 + \beta_2 H(xy)_2}.$$

Demonstrasse autem nobis videmur ad ξ vel $P(xy)$ significatione non mutata constantes addi posse, unde sequitur

$$\frac{\alpha_1 H_1 + \alpha_2 H_2}{\beta_1 H_1 + \beta_2 H_2} - \frac{\alpha_2}{\beta_2} = \frac{1}{\beta_2} \frac{(\alpha_1 \beta_2 - \alpha_2 \beta_1) H_1}{\beta_1 H_1 + \beta_2 H_2}$$

atque aeque obtinemus

$$\frac{\beta_1 H_1 + \beta_2 H_2}{H_1}$$

atque tandem

$$\frac{H_1}{H_2} \quad \text{et} \quad \frac{H_2}{H_1}.$$

Licet igitur in hoc casu duo $H(xy)$ arbitraria inter se dividi, ut ad ξ quoddam perveniatur. Cum vero pro $\varrho = 2$ semper functiones secundi ordinis existant (quia $\frac{\varrho + 2}{2} = 2$), apparet, si pro $f(xy)$ est $\varrho = 2$, duo $H(xy)$ factorem ordinis $(n-2)^{ti}$ habere, si ipsa sunt ordinis n^{ti}.

Vidimus supra § 4 extremo proprietatem illam functionum quarundam, ut ex duobus $H(xy)$, e. g. ex $H(xy)_1$ et $H(xy)_\varrho$ alia omnia formula

$$H(xy)_\lambda = \sqrt[\varrho-1]{H(xy)_1^{\varrho-\lambda}\; H(xy)_\varrho^{\lambda-1}} \tag{7}$$

deriventur.

Qua conditione impleta atque $H(xy)_1$ et $H(xy)_\varrho$ datis, valorem $x_1\, y_1$ eligere ita possumus, ut sit

$$\frac{H'(x_1y_1)_1}{H(x_1y_1)_1} = \frac{H'(x_1y_1)_\varrho}{H(x_1y_1)_\varrho}. \tag{8}$$

Sunt autem pro $k = 2$ conditiones (6′)

$$\frac{H'(xy)_1}{H(xy)_1} = \frac{H'(xy)_2}{H(xy)_2} = \ldots. = \frac{H'(xy)_\varrho}{H(xy)_\varrho}, \tag{9}$$

quae facile in hanc aliam formam rediguntur, ut

$$\frac{d}{dx}\left(\frac{H(xy)_2}{H(xy)_1}\right) = 0; \quad \frac{d}{dx}\left(\frac{H(xy)_3}{H(xy)_1}\right) = 0; \ldots. \tag{10}$$

habeant radicem communem.

Nunc sequitur ex (7)

$$\begin{aligned}\frac{H'(xy)_\lambda}{H(xy)_\lambda} &= d \lg H(xy)_\lambda \\ &= d \lg \sqrt[\varrho-1]{H(xy)_1^{\varrho-\lambda}\; H(xy)_\varrho^{\lambda-1}} \\ &= \frac{\varrho-\lambda}{\varrho-1}\,\frac{H'(xy)_1}{H(xy)_1} + \frac{\lambda-1}{\varrho-1}\,\frac{H'(xy)_\varrho}{H(xy)_\varrho},\end{aligned}$$

atque ob (8)

$$\frac{H'(x_1y_1)_\lambda}{H(x_1y_1)_\lambda} = \frac{(\varrho-\lambda)+(\lambda-1)}{\varrho-1}\,\frac{H'(x_1y_1)_1}{H(x_1y_1)_1} = \frac{H'(x_1y_1)_1}{H(x_1y_1)_1}.$$

Et quia idem locum habet pro omnibus $H(xy)$, valet conditio (9) ac formari potest functio secundi ordinis, si functiones $H(xy)$ repraesentant seriem geometricam.

Sit igitur

$$\begin{aligned}H(xy)_2 &= \Theta(xy)\, H(xy)_1 \\ &\cdots\cdots\cdots \\ H(xy)_\varrho &= \Theta^{\varrho-1}(xy)\, H(xy)_1;\end{aligned}$$

atque erit

$$\begin{aligned}P(xy) &= \frac{c_1 H(xy)_1 + c_2 H(xy)_2 + \ldots. + c_\varrho H(xy)_\varrho}{d_1 H(xy)_1 + d_2 H(xy)_2} \\ &= \frac{c_1 + c_2\Theta(xy) + \ldots. + c_\varrho \Theta^{\varrho-1}(xy)}{d_1 + d_2\Theta(xy)}.\end{aligned}$$

Quod nihil aliud docet, quam numerum q esse egalem 2, $c_3 = c_4 = \ldots. = c_\varrho = 0$, et esse

$$P(xy) = \frac{c_1 + c_2\Theta(xy)}{d_1 + d_2\Theta(xy)}.$$

Ipsum $\Theta(xy)$ igitur secundi ordinis est.

§ 6.

Proposuimus nobis invenire functiones $y^n = R(x)$, quae in formam $\eta^2 = R_1(\xi)$ possunt transformari, dedimusque § 5 conditiones necessarias. Ibidem vidimus, pro $\varrho = 1$ et $\varrho = 2$ semper esse impletas conditiones. Viri illustrissimi Briot et Bouquet cum jam tractavissent casum $\varrho = 1$, nunc nobis alius pro $\varrho = 2$, expeditius est disserendus.

Ac primum quidem, cum functionem $\xi = P(xy)$ formare sciamus, quae est quotiens duorum $H(xy)$ ad arbitrium sumptorum, necesse est requirantur etiam formulae $\eta = Q(xy)$. Hic autem aliam atque supra ineamus rationem. Etenim constat functionem integralem primi ordinis transformatione rationali non esse privatam hac proprietate, idemque existit in differentialibus integralium primi ordinis i. e. in $H(xy)dx$.

Est igitur:
$$H(xy)\,dx = \left(c_1 + c_2\xi + \ldots + c_\varrho \xi^{\varrho-1}\right)\frac{d\xi}{\eta}$$
$$\eta = \left(c_1 + \ldots + c_\varrho \xi^{\varrho-1}\right)\frac{d\xi}{H(xy)\,dx}$$
$$= Q(xy);$$
inde eliminatione variabilium x et y inter $f(xy) = 0$, $\xi = P(xy)$, $\eta = Q(xy)$ sequitur
$$\varphi(\xi\eta) = 0,$$
atque transformatio erit aequivalens, si contra x et y rationaliter per ξ et η exprimi possunt.

Etiam alia ratione $Q(xy)$ invenitur. Etenim, quoniam hic
$$\left(\xi - H(xy_1)\right)\left(\xi - H(xy_2)\right)\ldots\left(\xi - H(xy_n)\right)$$
$$= \xi^n - H^n(xy),$$
erit
$$\xi^n = \frac{(1)^{\lambda_1 n}\ldots(r)^{\lambda_r n}}{(1)^{m_1\lambda}\ldots(r)^{m_r\lambda}}$$
(3) $$= \frac{(\alpha)\,(\beta)}{(\gamma)\,(\delta)},$$
ubi α potest esse $= \beta$, $\gamma = \delta$. In transformatione uniformi, si invenimus ex (3) $x = F(\xi)$, F radicem secundum ex functione ordinis $2\varrho + 2$ aut $2\varrho + 1$ habebit, quam esse η constituamus, ita ut ex $F(\xi)$ ipsa inter ξ et η aequatio sequatur.

Liceat nunc enumerare pro $\varrho = 2$ formas et transformandas et transformatas, formulas pro x, y; ξ, η, atque simul differentialia integralium tertii ordinis functionum i. e. $H(xy, x'y')$, quarum supra enumeravimus proprietates insignes.

1. $y^{10} = (1)\,(2)^4\,(3)^5.$
$$H(xy, x'y') = \frac{y'^2\left(R(x)\,[(2) + (3)] - \tfrac{1}{5}R'(x)\,(2)\,(3)\right)}{2R(x)\,[y^2(2')\,(3') - y'^2(2)\,(3)]}\cdot$$
$$\left\{\frac{\left((2')\,(3') - y'^2\alpha\right)\left((2')\,(3') - y'^2\beta\right)y'(2)^2(3)^3}{\left((2)\,(3) - y^2\alpha\right)\left((2)\,(3) - y^2\beta\right)y(2')^2(3')^3} + 1\right\};$$

$$\xi = \frac{(2)\,(3)}{y^2}; \qquad \eta = \sqrt{a_1 - a_2}\,\frac{(2)^2\,(3)^3}{y^5};$$

$$x = \frac{a_1\,\xi^5 - a_2}{\xi^5 - 1}; \qquad y = \sqrt{a_1 - a_2}\,\frac{\xi^2\eta}{\xi^5 - 1};$$

$$\eta^2 = (a_1 - a_3)\,\xi^5 - (a_2 - a_3).$$

2. $y^8 = (1)\,(2)^3\,(3)^4;$

$$H(xy, x'y') = \frac{y'^2\big(R(x)\,[(2) + (3)] - \tfrac{1}{4}\,R'(x)\,(2)\,(3)\big)}{2\,R(x)\,[y^2\,(2')\,(3') - y'^2\,(2)\,(3)]}.$$

$$\left\{\frac{\big((2')\,(3') - y'^2\alpha\big)\,\big((2')\,(3') - y'^2\beta\big)\,y'\,(2)^2\,(3)^3}{\big((2)\,(3) - y^2\alpha\big)\,\big((2)\,(3) - y^2\beta\big)\,y\,(2)^2\,(3)^3} + 1\right\};$$

$$\xi = \frac{(2)\,(3)}{y^2}; \qquad \eta = \sqrt{a_1 - a_2}\,\frac{(2)^2\,(3)^3}{y^5};$$

$$x = \frac{a_1\,\xi^4 - a_2}{\xi^4 - 1}; \qquad y = \sqrt{a_1 - a_2}\,\frac{\xi\eta}{\xi^4 - 1};$$

$$\eta^2 = \xi\,\big((a_1 - a_3)\,\xi^4 - (a_2 - a_3)\big).$$

3. $y^6 = (1)\,(2)\,(3)^4;$

$$H(xy, x'y') = \frac{y'\,\big(R(x) - \tfrac{1}{6}\,R'(x)\,(3)\big)}{2\,R(x)\,[y\,(3') - y'\,(3)]}.$$

$$\left\{\frac{\big((3') - \alpha y'\big)\,\big((3') - \beta y'\big)\,\Big(2\,y\Big[\frac{(3)^2}{(1)\,(2)} - 1\Big] - (a_1 + a_2 - 2\,a_3)\Big)\,y'\,(3')^3}{\big((3) - \alpha y\big)\,\big((3) - \beta y\big)\,\Big(2y'\Big[\frac{(3')^2}{(1')\,(2')} - 1\Big] - (a_1 + a_2 - 2\,a_3)\Big)\,y\,(3)^3} + 1\right\};$$

$$\xi = \frac{(3)}{y}; \qquad \eta = \Big(2\,y\,\Big[\frac{(3)^2}{(1)\,(2)} - 1\Big] - (a_1 + a_2 - 2\,a_3)\Big)\,\frac{(3)^3}{y^3};$$

$$x = \frac{(a_1 + a_2)\,\xi^6 - 2\,a_3 + \xi^3\eta}{2\,(\xi^6 - 1)}; \quad y = \frac{(a_1 + a_2 - 2\,a_3)\,\xi^3 + \eta}{2\,\xi^3\,(\xi^6 - 1)};$$

$$\eta^2 = (a_1 - a_2)^2\,(\xi^6 - 1) - (a_1 + a_2 - 2\,a_3)^2.$$

4. $y^5 = (1)\,(2)\,(3)^3;$

$$H(xy, x'y') = \frac{y'\,\big(R(x) - \tfrac{1}{5}\,R'(x)\,(3)\big)}{2\,R(x)\,[y\,(3') - y'\,(3)]}.$$

$$\left\{\frac{\big((3') - \alpha y'\big)\,\big((3') - \beta y'\big)\,\Big(2\,y\,\Big[\frac{(3)^2}{(1)\,(2)} - 1\Big] - (a_1 + a_2 - 2\,a_3)\Big)\,y'\,(3')^3}{\big((3) - \alpha y\big)\,\big((3) - \beta y\big)\,\Big(2y'\,\Big[\frac{(3')^2}{(1')\,(2')} - 1\Big] - (a_1 + a_2 - 2\,a_3)\Big)\,y\,(3)^3} + 1\right\};$$

$$\xi = \frac{(3)}{y}; \qquad \eta = \Big(2\,y\,\Big[\frac{(3)^2}{(1)\,(2)} - 1\Big] - (a_2 + a_2 - 2\,a_3)\Big)\,\frac{(3)^3}{y^3};$$

$$x = \frac{(a_1 + a_2)\,\xi^5 - 2\,a_3 + \xi^2 \eta}{2\,(\xi^5 - 1)}\,; \qquad y = \frac{(a_1 + a_2 - 2\,a_3)\,\xi^3 + \eta}{2\,\xi^3\,(\xi^5 - 1)}\,;$$

$$\eta^2 = \xi\,\big((a_1 - a_2)^2\,(\xi^5 - 1) + (a_1 + a_2 - 2\,a_3)^2\big).$$

5. $y^6 = (1)^2\,(2)^3\,(3)^3\,(4)^4$;

$$H(xy,\,x'y') = \frac{y'^{\,2}\left(R\,(x)\,\dfrac{d}{d\,x}\sqrt[3]{\dfrac{R\,(x)\,(1)}{(4)}} - \dfrac{1}{3}\,R'\,(x)\sqrt[3]{\dfrac{R\,(x)\,(1)}{(4)}}\right)}{2\,R\,(x)\left[y^2\sqrt[3]{\dfrac{R\,(x')(1')}{(4')}} - y'^{\,2}\sqrt[3]{\dfrac{R\,(x)\,(1)}{(4)}}\right]}\,.$$

$$\left\{\frac{\left(\sqrt[3]{\dfrac{R\,(x')\,(1')}{(4')}} - \alpha y'^{\,2}\right)\left(\sqrt[3]{\dfrac{R\,(x')\,(1')}{(4')}} - \beta y'^{\,2}\right)(1)\,(2)\,(3)\,.\,y^3}{\left(\sqrt[3]{\dfrac{R\,(x)\,(1)}{(4)}} - \alpha y^2\right)\left(\sqrt[3]{\dfrac{R\,(x)\,(1)}{(4)}} - \beta y^2\right)(1')\,(2')\,(3')\,.\,y'^{\,3}} + 1\right\};$$

$$\xi = \frac{(1)\,(2)\,(3)\,(4)}{y^2}\,; \qquad \eta = (a_4 - a_1)\,\frac{(1)\,(2)\,(3)}{y}\,;$$

$$x = \frac{a_4\,\xi^3 - a_1}{\xi^3 - 1}\,; \qquad y = (a_4 - a_1)\,\frac{\xi \eta}{\xi^3 - 1}\,;$$

$$\eta^2 = \big((a_4 - a_2)\,\xi^3 - (a_1 - a_2)\big)\big((a_4 - a_3)\,\xi^3 - (a_1 - a_3)\big).$$

6. $y^4 = (1)\,(2)^2\,(3)^2\,(4)^3$;

$$H(xy,\,x'y') = \frac{y'^{\,2}\left(R\,(x)\,\dfrac{d}{d\,x}\sqrt[2]{\dfrac{R\,(x)\,(4)}{(1)}} - \dfrac{1}{2}\,R'\,(x)\sqrt[2]{\dfrac{R\,(x)\,(4)}{(1)}}\right)}{2\,R\,(x)\left[y^2\sqrt[2]{\dfrac{R\,(x')(4')}{(1')}} - y'^{\,2}\sqrt[2]{\dfrac{R\,(x)\,(4)}{(1)}}\right]}\,.$$

$$\left\{\frac{\left(\sqrt[2]{\dfrac{R\,(x')\,(4')}{(1')}} - \alpha y'^{\,2}\right)\left(\sqrt[2]{\dfrac{R\,(x')\,(4')}{(1')}} - \beta y'^{\,2}\right)(2)\,(3)\,(4)\;y^3}{\left(\sqrt[2]{\dfrac{R\,(x)\,(4)}{(1)}} - \alpha y'^{\,2}\right)\left(\sqrt[2]{\dfrac{R\,(x)\,(4)}{(1)}} - \beta y^2\right)(2')\,(3')\,(4')\;y'^{\,3}} + 1\right\};$$

$$\xi = \frac{(2)\,(3)\,(4)^2}{y^2}\,; \qquad \eta = (a_1 - a_4)\,\frac{(2)\,(3)\,(4)}{y}\,;$$

$$x = \frac{a_4\,\xi^3 - a_1}{\xi^3 - 1}\,; \qquad y = (a_1 - a_4)\,\frac{\xi \eta}{\xi^2 - 1}\,;$$

$$\eta^2 = \xi\big((a_1 - a_2)\,\xi^2 - (a_4 - a_2)\big)\big((a_1 - a_3)\,\xi^2 - (a_4 - a_3)\big).$$

7. $y^3 = (1)\,(2)\,(3)^2\,(4)^2$;

$$H\,(xy,\,x'y') = \frac{y'\,\big(R(x)\,[(3) + (4)] - \tfrac{1}{3}\,R'\,(x)\,(3)\,(4)\big)}{2\,R\,(x)\,[y\,(3')\,(4') - y'\,(3)\,(4)]}\,.$$

$$\left\{\frac{\big((3')(4')-\alpha y'\big)\big((3')(4')-\beta y'\big)\,y^2\left(\frac{(3)}{(1)}\frac{(4)}{(2)}(a_1+a_2-2x)-(a_3+a_4-2x)\right)}{\big((3)(4)-\alpha y\big)\big((3)(4)-\beta y\big)\,y'^2\left(\frac{(3')}{(1')}\frac{(4')}{(2')}(a_1+a_2-2x')-(a_3+a_4-2x')\right)}+1\right\};$$

$$\xi=\frac{(3)(4)}{y}; \qquad \eta=\frac{(3)(4)}{(1)(2)}(a_1+a_2-2x)-(a_3+a_4-2x);$$

$$x=\frac{(a_3+a_4)-(a_1+a_2)\,\xi^3+\eta}{2(1-\xi^3)};$$

$$y=\frac{4\xi^3(1-\xi^3)^2}{\big(\eta-\xi^3(a_1+a_2-2a_3)+(a_4-a_3)\big)\big(\eta-\xi^3(a_1+a_2-2a_4)+(a_3-a_4)\big)};$$

$$\eta^2=\big(\xi^3(a_1+a_2)-(a_3+a_4)\big)^2-4(1-\xi^3)(a_3a_4-a_1a_2\xi^3).$$

VITA.

Natus sum Eugenius Netto Halis Saxorum die XXX m. Jun. a. h. s. XLVI patre Henrico. matre Sophia e gente Neumann, quos parentes vivos maxime veneror. Primis literarum elementis Halis imbutus, puer undecim annorum gymnasium adii Fridericum-Guilelmianum Berolini, auspiciis florens adhuc humanissimi Ranke, ubi viri Ill. Luchterhandt et Schellbach ad studia mathematica vocabant adolescentis animum. Maturitatis testimonio instructus, a. h. s. LXVI a rectore Magnif. Braun civibus universitatis Fridericae Guilelmae Berolinensis adscriptus apud Spectat. philosophorum decanum, virum Ill. Kummer nomen professum sum. Seminarii mathematici aditus ab secundo semestri mihi patuit; excercitationibus per sex semestria interfui. Per octo semestria disserentes audivi viros Ill. Auvers, Dove, Foerster, Harms, Kronecker, Kummer, Quincke, Thomé, Trendelenburg, Weierstrass. Quibus viris omnibus optime de me meritis, praecipue viris Ill. Kronecker, Kummer, Weierstrass summas ago gratias semperque gratum habebo animum.

THESES.

1. Facilis in analysin valorum negativorum introductio nocuit cognitioni valorum complexorum.
2. Generalitatem atque necessitatem signa esse judiciorum a priori, Kantius non satis confirmavit.
3. Trigonometria ita est docenda, ut a circulo non a triangulo rectangulo considerando initium capiatur.